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Uniqueness of norm on $L^p(G)$ and $C(G)$ when G is a compact group

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Abstract

Let G be a compact group. If the trivial representation of G is not weakly contained in the left regular representation of G on $L_0^2(G)$ and X is either $L^p(G)$ for $1<p\leqslant\infty$ or $C(G)$, then we show that every complete norm $|\cdot|$ on X that makes translations from $(X,|\cdot|)$ into itself continuous is equivalent to $\|\cdot\|_p$ or $\|\cdot\|_\infty$ respectively. If $1<p\leqslant\infty$ and every left invariant linear functional on $L^p(G)$ is a constant multiple of the Haar integral, then we show that every complete norm $|\cdot|$ on $L^p(G)$ that makes translations from $(L^p(G),|\cdot|)$ into itself continuous and that makes the map $t\mapsto L_t$ from G into $\mathscr{L}(L^p(G),|\cdot|)$ bounded is equivalent to $\|\cdot\|_p$.

Keywords: Uniqueness of norm; Mean-zero weak containment property; Translation invariant functionals

1. Introduction

The uniqueness-of-norm problem is a classical topic in automatic continuity theory which has been mainly developed in the context of Banach algebras. The most important result in this area is the famous Johnson theorem [6] that every semisimple Banach algebra $(A,\|\cdot\|)$ carries a unique Banach algebra topology. This entails that

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every complete norm $|\cdot|$ on A that makes multiplication operators from $(A, |\cdot|)$ into itself continuous is automatically equivalent to $||\cdot||$.

It has been recently shown by Jarosz [5] that every complete norm $|\cdot|$ on $L^p(\mathbb{T})$ for $1<p<\infty$ that makes translations from $(L^p(\mathbb{T}), |\cdot|)$ continuous is equivalent to $||\cdot||_p$. This result has been extended in [3] to $L^p(G)$ for any compact and connected abelian group G. In this paper we extend Jarosz result to a wide variety of nonabelian compact groups by proving that if the trivial representation of a compact group G is not weakly contained in the left regular representation of G on $L^2_0(G)$ and X is either $L^p(G)$ for $1<p\leqslant\infty$ or $C(G)$, then the norm $||\cdot||_p$ is the only complete norm on X that makes all translations continuous. This class of groups includes the matrix groups $SO(n)$ for $n\geqslant 3$ (note that $SO(2)=\mathbb{T}$ and hence [5] applies in this case) and all compact simple Lie groups. It is known [12,16] that for this class of groups every left invariant linear functional on X is continuous. In fact, it turns out that the uniqueness of translation invariant norms on $L^p(G)$ is closely related to the classical problem whether or not every translation invariant linear functional is automatically continuous. As a matter of fact, we show that if $1<p\leqslant\infty$ and if the compact group G is such that every left invariant linear functional on $L^p(G)$ is continuous, then every complete norm $|\cdot|$ on $L^p(G)$ that makes translations from $(L^p(G), |\cdot|)$ into itself continuous and that makes the map $t\mapsto L_t$ from G into $\mathscr{L}(L^p(G), |\cdot|)$ bounded is equivalent to $||\cdot||_p$. It should be pointed out that the former condition is fulfilled in the case when the map $t\mapsto L_t f$ from G into $(L^p(G), |\cdot|)$ is continuous for each $f\in L^p(G)$. Moreover we show that if X is either $L^p(G)$ for $1\leqslant p\leqslant\infty$ or $C(G)$ and if there exists a discontinuous two-sided invariant linear functional on X, then there exists a complete norm $|\cdot|$ on X that makes translations from $(X, |\cdot|)$ into itself continuous, that makes the maps $t\mapsto L_t$ and $t\mapsto R_t$ from G into $\mathscr{L}(X, |\cdot|)$ bounded, and that is not equivalent to $||\cdot||$. There is a number of authors who have made substantial contributions to automatic continuity of translation invariant linear functionals. For an excellent survey of those obtained prior to 1981 we refer the reader to [9]. Some results obtained since then are quoted here.

2. Preliminaries

A unitary representation of a locally compact group G is a strongly continuous homomorphism π of G into the group of unitary operators on some nonzero Hilbert space H_π. A subspace M of H_π is said to be π-invariant if $\pi(t)(M)\subset M$ for each $t\in G$. π is said to be irreducible if there are no nontrivial closed π-invariant subspaces of H_π. Two unitary representation π and π' are said to be equivalent if there is a unitary operator $U: H_\pi\to H_{\pi'}$ such that $U\pi(t)=\pi'(t)U$ for each $t\in G$. We shall denote by $\hat{G}$ the set of equivalence classes of irreducible unitary representations of G and we shall denote by $[\pi]$ the class of an irreducible unitary representation π of G.

Throughout the paper, G stands for a compact group and $(X, ||\cdot||)$ is either $(L^p(G), ||\cdot||_p)$ for $1\leqslant p\leqslant\infty$ or $(C(G), ||\cdot||_\infty)$. We assume that the Haar measure λ on G is normalized so that $\lambda(G)=1$. It is well-known that every irreducible unitary representation of G is finite dimensional. Every unitary representation π of G determines a continuous operator, still denoted by π, from X into $\mathscr{L}(H_\pi)$ which is defined by

$$\pi(f) = \int_G f(t)\pi(t)\,dt$$

for each $f \in X$. It is evident that $\pi(X) \neq \{0\}$. For every $t \in G$ we define the left and right translation operators L_t and R_t from X onto itself by

$$(L_t f)(x) = f(t^{-1}x) \quad \text{and} \quad (R_t f)(x) = f(xt)$$

for all $f \in X$ and $x \in G$. Let $\mathfrak{A}$ denote the subalgebra of the algebra of all linear operators from X into itself generated by all the right translation operators on X. A standard fact we shall use is that, for $t \in G$ and $f \in X$, we have

$$\pi(R_t f) = \pi(f)\pi(t^{-1}). \tag{1}$$

Moreover, we have

$$\pi(L_t f) = \pi(t)\pi(f).$$

For Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$ we shall denote by $\mathscr{L}(\mathfrak{X}, \mathfrak{Y})$ the Banach space of all continuous operators from $\mathfrak{X}$ into $\mathfrak{Y}$ (we shall write $\mathscr{L}(\mathfrak{X})$ instead of $\mathscr{L}(\mathfrak{X}, \mathfrak{X})$). For simplicity of notation, we shall denote by $\|\cdot\|$ the operator norm of $\mathscr{L}(\mathfrak{X}, \mathfrak{Y})$ irrespective of $\mathfrak{X}$ and $\mathfrak{Y}$. A key notion to study the continuity of a linear map Φ from $\mathfrak{X}$ into $\mathfrak{Y}$ is that of the separating space $\mathfrak{S}(\Phi)$ of Φ which is defined as follows:

$$\mathfrak{S}(\Phi) = \{y \in \mathfrak{Y} : \text{ there exists } (x_n) \to 0 \text{ in } \mathfrak{X} \text{ with } (\Phi(x_n)) \to y\}.$$

The separating space is a closed subspace of $\mathfrak{Y}$ that measures the closedness of Φ and the closed graph theorem shows that Φ is continuous if and only if $\mathfrak{S}(\Phi) = \{0\}$. For a thorough discussion of the separating space we refer the reader to [14].

In the sequel X will be assumed to be equipped with a complete norm $|\cdot|$ that makes translations from $(X, |\cdot|)$ into itself continuous. Therefore

$$\mathfrak{A} \subset \mathscr{L}(X, \|\cdot\|) \cap \mathscr{L}(X, |\cdot|). \tag{2}$$

We shall denote by i_X the identity map from $(X, |\cdot|)$ onto $(X, \|\cdot\|)$, and we shall abbreviate $\mathfrak{S}(i_X)$ to $\mathfrak{S}$. It should be noted that, on account of the open mapping theorem, $|\cdot|$ is equivalent to $\|\cdot\|$ if and only if i_X is continuous and this latter condition is equivalent to $\mathfrak{S} = \{0\}$. In order to check whether or not $\mathfrak{S} = \{0\}$ we shall consider the subset Δ of $\hat{G}$ defined by

$$\Delta = \{[\pi] \in \hat{G} : \pi(\mathfrak{S}) \neq \{0\}\}.$$

This definition makes sense because if $\pi' \in [\pi]$ and $U : H_\pi \to H_{\pi'}$ is a unitary operator such that $U\pi(t) = \pi'(t)U$ for each $t \in G$, then $U\pi(f) = \pi'(f)U$ for each $f \in X$ and so

$\pi(\mathfrak{S}) = \{0\}$ if and only if $\pi'(\mathfrak{S}) = \{0\}$. We now gather together a few properties that we shall use in the sequel.

Lemma 2.1. *The following assertions hold*:

(i) $\mathfrak{S}$ *is translation invariant.*
(ii) $\mathfrak{S}(i_X^{-1}) = \mathfrak{S}$.
(iii) *Let* $f \in X$ *be such that* $\pi(f) = 0$ *for each* $[\pi] \in \hat{G}$. *Then* $f = 0$.
(iv) $|\cdot|$ *is equivalent to* $\|\cdot\|$ *if and only if* $\Delta = \emptyset$.

Proof. The proof of first and second assertions are just a straightforward verification.

We now suppose that f satisfies the hypothesis of (iii). Then $\int_G f(t)\tau(t)\,dt = 0$ for each trigonometric polynomial τ. Since trigonometric polynomials are dense in $C(G)$ with the uniform norm, it may be concluded that $\int_G f(t)g(t)\,dt = 0$ for each $g \in C(G)$ which yields $f = 0$.

Finally, from the third assertion it follows immediately that $\Delta = \emptyset$ if and only if $\mathfrak{S} = \{0\}$ and this latter condition is equivalent to the property that $|\cdot|$ is equivalent to $\|\cdot\|$. $\square$

Lemma 2.2. *Let* $\mathfrak{X}$, $\mathfrak{Y}$, *and* $\mathfrak{Z}$ *be Banach spaces and let* Φ *a linear map from* $\mathfrak{X}$ *into* $\mathfrak{Y}$. *Then there exists a constant* D_1 *such that* $\|\Psi\Phi\| \leqslant D_1\|\Psi\|$ *for each* $\Psi \in \mathscr{L}(\mathfrak{Y}, \mathfrak{Z})$ *such that* $\Psi\Phi$ *is continuous.*

Proof. Set $\mathfrak{U} = \{\Psi \in \mathscr{L}(\mathfrak{Y}, \mathfrak{Z}) : \Psi\Phi \text{ is continuous}\}$. From the Banach–Steinhaus theorem it follows that $\mathfrak{U}$ is closed in $\mathscr{L}(\mathfrak{Y}, \mathfrak{Z})$. On the other hand, from the closed graph theorem it may be concluded that the map $\Psi \mapsto \Psi\Phi$ from $\mathfrak{U}$ into $\mathscr{L}(\mathfrak{X}, \mathfrak{Z})$ is continuous. Certainly, this implies the truthfulness of the lemma. $\square$

Our method of proof involves the so-called gliding hump argument. This is a basic principle in automatic continuity and for a recent account of this theory we refer the reader to [1]. The following result illustrates this technique.

Lemma 2.3. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be Banach spaces and let* Φ *be a linear map from* $\mathfrak{X}$ *into* $\mathfrak{Y}$. *Suppose that there exist* $T_n \in \mathscr{L}(\mathfrak{X})$ *and continuous linear operators* S_n *from* $\mathfrak{Y}$ *into Banach spaces* $\mathfrak{Y}_n$ *such that* $S_n\Phi T_1 \cdots T_m$ *is continuous for* $m > n$, *then* $S_n\Phi T_1 \cdots T_n$ *is continuous for sufficiently large* n.

3. Gliding hump sequences

Needless to say that we shall apply Lemma 2.3 with $\Phi = i_X$. Of course, it is required to construct the appropriate sequences (T_n) and (S_n) of continuous operators. This is just the purpose of this section. It should be pointed out that the main difficulty in carrying out this construction is the following. When dealing with translation invariant norms (unlike algebra norms) we are concerned with the noncomplete algebra $\mathfrak{A}$ and therefore the classical approach, such as in [7], does not work here.

Lemma 3.1. *Let π be an irreducible unitary representation of G, let Y be a right invariant linear subspace of X, and let $x \in H_\pi \backslash \{0\}$. If $\pi(Y)(x) = \{0\}$, then $\pi(Y) = \{0\}$.*

Proof. Set $M = \{y \in H_\pi : \pi(f)(y) = 0 \text{ for each } f \in Y\}$. Obviously $x \in M$ and it is easily seen that M is a linear subspace of H_π. Let $y \in M$ and $t \in G$. For every $f \in Y$ we have $R_{t^{-1}}(f) \in Y$ and taking into account (1) we conclude that $\pi(f)\pi(t)(y) = \pi(R_{t^{-1}}(f))(y) = 0$. Consequently, M is π-invariant. Since π is irreducible, it follows that $M = H_\pi$ which yields $\pi(Y) = \{0\}$. $\square$

We verify next that each irreducible unitary representation of G extends by linearity to a representation of $\mathfrak{A}$.

Lemma 3.2. *Let π be an irreducible unitary representation of G. Then there is a linear map π from $\mathfrak{A}$ into $\mathscr{L}(H_\pi)$ that satisfies the following properties*:

(i) $\pi(T(f)) = \pi(f)\pi(T)$ *for all* $f \in X$ *and* $T \in \mathfrak{A}$.
(ii) $\pi(R_t) = \pi(t)^{-1}$ *for each* $t \in G$.
(iii) $\pi(ST) = \pi(T)\pi(S)$ *for all* $S, T \in \mathfrak{A}$.

Moreover if $\pi' \in [\pi]$ and $U : H_\pi \to H_{\pi'}$ is a unitary operator such that $U\pi(t) = \pi'(t)U$ for each $t \in G$, then $U\pi(T) = \pi'(T)U$ for each $T \in \mathfrak{A}$.

Proof. Let $T \in \mathfrak{A}$ and suppose that there are $\Gamma, \Gamma' \in \mathscr{L}(H_\pi)$ such that $\pi(T(f)) = \pi(f)\Gamma = \pi(f)\Gamma'$ for each $f \in L^1(G)$. For every $x \in H_\pi$ we have $\pi(f)(\Gamma - \Gamma')(x) = 0$ for each $f \in X$ and the preceding lemma now shows that $(\Gamma - \Gamma')(x) = 0$. Thus $\Gamma = \Gamma'$.

Let $\mathfrak{B}$ be the linear subspace of $\mathfrak{A}$ which consists of all $T \in \mathfrak{A}$ for which $\pi(T)$ can be defined. According to (1), $R_t \in \mathfrak{B}$ and $\pi(R_t) = \pi(t)^{-1}$ for each $t \in G$. If $S, T \in \mathfrak{B}$, then

$$\pi((ST)(f)) = \pi(S(T(f))) = \pi(T(f))\pi(S) = \pi(f)\pi(T)\pi(S)$$

for each $f \in X$. Therefore $ST \in \mathfrak{B}$ and $\pi(ST) = \pi(T)\pi(S)$. Hence $\mathfrak{B}$ is a subalgebra of $\mathscr{L}(X)$ that contains all the right translation operators and so $\mathfrak{B} = \mathfrak{A}$.

Finally, for all $f \in X$ and $T \in \mathfrak{A}$ we have $U\pi(T(f)) = \pi'(T(f))U = \pi'(f)\pi'(T)U$ and $U\pi(T(f)) = U\pi(f)\pi(T) = \pi'(f)U\pi(T)$. Hence $\pi'(T)U = U\pi(T)$. $\square$

Lemma 3.3. *Let π and π' be nonequivalent irreducible unitary representations of G. If $x \in H_\pi \backslash \{0\}$ and $y \in H_{\pi'} \backslash \{0\}$, then there exists $T \in \mathfrak{A}$ such that $\pi(T)(x) = 0$ and $\pi'(T)(y) \neq 0$.*

Proof. Let $M = \{\pi(T)(x) : T \in \mathfrak{A}\}$. It is clear that $x = \pi(R_e)(x)$ and so that $x \in M$. We claim that M is π-invariant. Indeed, set $u \in M$, $t \in G$, and write $u = \pi(T)(x)$ for some $T \in \mathfrak{A}$. We have

$$\pi(t)(u) = \pi(R_{t^{-1}})\pi(T)(x) = \pi(TR_{t^{-1}})(x) \in M.$$

Since π is irreducible, it follows that $M = H_\pi$.

Suppose the lemma is false. Then we can define a (bounded) linear map $\Gamma : H_\pi \to H_{\pi'}$ by $\Gamma(\pi(T)(x)) = \pi'(T)(y)$. We note that $\Gamma(x) = \Gamma(\pi(R_e)(x)) = \pi'(R_e)(y) = y \neq 0$. We now proceed to show that $\Gamma\pi(t) = \pi'(t)\Gamma$ for each $t \in G$. Indeed, we have

$$\begin{aligned} \Gamma\pi(t)(\pi(T)(x)) &= \Gamma\pi(R_{t^{-1}})(\pi(T)(x)) = \Gamma(\pi(TR_{t^{-1}})(x)) \\ &= \pi'(TR_{t^{-1}})(y) = \pi'(R_{t^{-1}})\pi'(T)(y) = \pi'(t)\Gamma(\pi(T)(x)) \end{aligned}$$

for all $T \in \mathfrak{A}$ and $t \in G$. By Schur's Lemma [4, Lemma 3.5] it follows that $\Gamma = 0$, which gives a contradiction. $\square$

Lemma 3.4. *Let Σ be an infinite set of pairwise nonequivalent irreducible unitary representations of G. Then one of the following assertions holds*:

(i) *There exist sequences (π_n) in Σ, (T_n) in $\mathfrak{A}$ and (x_n) with $x_n \in H_{\pi_n}$ such that $\pi_n(T_n)\cdots\pi_n(T_1)(x_n) \neq 0$ and $\pi_n(T_{n+1})\cdots\pi_n(T_1)(x_n) = 0$ for each $n \in \mathbb{N}$.*

(ii) *There exists $T \in \mathfrak{A}$ such that the set $\{\pi \in \Sigma : \pi(T) \neq 0\}$ is nonempty and finite.*

Proof. Suppose that the second assertion does not hold. Set $\pi_1 \in \Sigma$, $T_1 = R_e$, and $x_1 \in H_{\pi_1}\setminus\{0\}$. Then $\pi_1(T_1)(x_1) = x_1 \neq 0$. Assume that $\pi_1, \ldots, \pi_n \in \Sigma$, $T_1, \ldots, T_n \in \mathfrak{A}$, and $x_1 \in H_{\pi_1}, \ldots, x_n \in H_{\pi_n}$ have been chosen so that $\pi_k(T_k)\cdots\pi_k(T_1)(x_k) \neq 0$ if $k \leqslant n$ and $\pi_k(T_{k+1})\cdots\pi_k(T_1)(x_k) = 0$ if $k < n$. Since $\pi_n(T_1\cdots T_n) = \pi_n(T_n)\cdots\pi_n(T_1) \neq 0$ and the second assertion fails, it follows that the set $\{\pi \in \Sigma : \pi(T_1\cdots T_n) \neq 0\}$ is infinite. Pick $\pi_{n+1} \in \Sigma\setminus\{\pi_n\}$ and $x_{n+1} \in H_{\pi_{n+1}}$ such that $\pi_{n+1}(T_1\cdots T_n)(x_{n+1}) \neq 0$. By Lemma 3.3 there exists $T_{n+1} \in \mathfrak{A}$ such that

$$\pi_n(T_{n+1})(\pi_n(T_n)\cdots\pi_n(T_1)(x_n)) = 0$$

and

$$\pi_{n+1}(T_{n+1})(\pi_{n+1}(T_n)\cdots\pi_{n+1}(T_1)(x_{n+1})) \neq 0.$$

The sequences (π_n), (T_n), and (x_n) constructed in this way satisfy the requirements of the first assertion. $\square$

On account of Lemma 3.2, if $T \in \mathfrak{A}$ and $\pi' \in [\pi] \in \hat{G}$, then $\pi(T) = 0$ if and only if $\pi'(T) = 0$. Therefore the definition of the set $\{[\pi] \in \Delta : \pi(T) \neq 0\}$ makes sense.

Lemma 3.5. *One of the following assertions holds*:

(i) *The set Δ is finite.*
(ii) *There exists $T \in \mathfrak{A}$ such that the set $\{[\pi] \in \Delta : \pi(T) \neq 0\}$ is nonempty and finite.*

Proof. Suppose that Δ is infinite and that there exist sequences (π_n), (T_n), and (x_n) as in the first assertion of Lemma 3.4. Let S_n be the continuous linear operator from $(X, ||\cdot||)$ into H_{π_n} defined by $S_n(f) = \pi_n(f)(x_n)$ for all $f \in X$ and $n \in \mathbb{N}$. It is easily checked that $S_n i_X T_1 \cdots T_m = 0$ whenever $m > n$. On account of Lemma 2.3 together with (2), there exists $n \in \mathbb{N}$ such that the operator

$$f \mapsto \pi_n((T_1 \cdots T_n)(f))(x_n) = \pi_n(f)(\pi_n(T_n) \cdots \pi_n(T_1)(x_n))$$

from $(X, |\cdot|)$ into H_{π_n} is continuous. Sinclair [14, Lemma 1.3] now shows that

$$\pi_n(\mathfrak{S})(\pi_n(T_n) \cdots \pi_n(T_1)(x_n)) = 0.$$

Since $\mathfrak{S}$ is translation invariant, Lemma 3.1 gives $\pi_n(\mathfrak{S}) = 0$, a contradiction.

According to Lemma 3.4, either Δ is finite and so (i) is obtained or there exists $T \in \mathfrak{A}$ such that $\{\pi \in \Delta : \pi(T) \neq 0\}$ is nonempty and finite and so (ii) holds. $\square$

4. Matrix trick

In order to put into action harmonic analysis in our problem we require to consider matrix-valued functions instead of complex-valued functions.

For each $n \in \mathbb{N}$, let $\mathscr{M}_n(X)$ denote the linear space of all $n \times n$ matrices with entries from X. Note that $\mathscr{M}_n(\mathbb{C})$ can be embedded into $\mathscr{M}_n(X)$ in the obvious way. For every $F = (f_{i,j}) \in \mathscr{M}_n(L^1(G))$ we define $\int_G F(t)\,dt$ to be the matrix $(\int_G f_{i,j}(t)\,dt) \in \mathscr{M}_n(\mathbb{C})$. It is obvious that

$$\int_G F(st)\,dt = \int_G F(t)\,dt \tag{3}$$

for each $s \in G$. For every linear operator T from X into itself, the map $(f_{i,j}) \mapsto (T(f_{i,j}))$ is a linear operator from $\mathscr{M}_n(X)$ into itself. For simplicity of notation, we continue to write T for this map.

$\mathscr{M}_n(X)$ is made into a Banach space by defining

$$||(f_{i,j})|| = \sum_{i=1}^{n} \sum_{j=1}^{n} ||f_{i,j}||$$

for each $(f_{i,j}) \in \mathscr{M}_n(X)$. In particular we define

$$||(f_{i,j})||_\infty = \sum_{i=1}^{n} \sum_{j=1}^{n} ||f_{i,j}||_\infty$$

and

$$||(a_{i,j})|| = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{i,j}|$$

for all $(f_{i,j}) \in \mathcal{M}_n(C(G))$ and $(a_{i,j}) \in \mathcal{M}_n(\mathbb{C})$. Moreover, $\mathcal{M}_n(X)$ turns into a Banach space when endowed with the norm

$$|(f_{i,j})| = \sum_{i=1}^{n} \sum_{j=1}^{n} |f_{i,j}|$$

for each $(f_{i,j}) \in \mathcal{M}_n(X)$. On the other hand, it should be pointed out that $C(G)X \subset X$ and then matrix multiplications FH and HF with $F \in \mathcal{M}_n(X)$ and $H \in \mathcal{M}_n(C(G))$ are natural.

Lemma 4.1. *Let* $F \in \mathcal{M}_n(X)$, $H \in \mathcal{M}_n(C(G))$, *and* $A \in \mathcal{M}_n(\mathbb{C})$. *Then the following assertions hold*:

(i) $HF, FH \in \mathcal{M}_n(X)$, $||HF|| \leqslant ||H||_\infty ||F||$, *and* $||FH|| \leqslant ||H||_\infty ||F||$.
(ii) $|AF| \leqslant ||A|| |F|$ *and* $|FA| \leqslant ||A|| |F|$.

Proof. We begin by observing that it is easy to check that $HF, FH \in \mathcal{M}_n(X)$.

We now write $F = (f_{i,j})$ and $H = (h_{i,j})$. Then

$$\begin{aligned} ||HF|| &= \sum_{i=1}^{n} \sum_{j=1}^{n} \left|\left| \sum_{k=1}^{n} h_{i,k} f_{k,j} \right|\right| \leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} ||h_{i,k} f_{k,j}|| \\ &\leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} ||h_{i,k}||_\infty ||f_{k,j}|| = \sum_{i=1}^{n} \sum_{k=1}^{n} ||h_{i,k}||_\infty \sum_{j=1}^{n} ||f_{k,j}|| \\ &\leqslant \sum_{i=1}^{n} \sum_{k=1}^{n} ||h_{i,k}||_\infty ||F|| = ||H||_\infty ||F||. \end{aligned}$$

In the same manner we can see that $||FH|| \leqslant ||H||_\infty ||F||$. The matrix functions AF and FA can be handled in much the same way, the only difference being at the end of the first line in which $|a_{i,k} f_{k,j}| = |a_{i,k}||f_{k,j}|$, where we write $A = (a_{i,j})$. $\square$

Let π be an n-dimensional unitary representation of G. When an orthonormal basis in H_π is fixed, $\pi(t)$ is represented by a (unitary) matrix $\pi_{\mathcal{M}}(t) \in \mathcal{M}_n(\mathbb{C})$ for each $t \in G$. From now on we think of $\pi_{\mathcal{M}}$ as being an element of $\mathcal{M}_n(C(G))$.

Throughout this section we assume that the group G is such that there exist $N \in \mathbb{N}$, a constant C_1, and a subset $K \subset G$ such that each function $f \in X$ has a representation of the form

$$f = a + \sum_{k=1}^{N} (f_k - L_{t_k}(f_k)),$$

where $a \in \mathbb{C}$, $t_k \in K$, $f_k \in X$, and $\|f_k\| \leqslant C_1 \|f\|$ for $k = 1, \ldots, N$.

Lemma 4.2. *Let π be an n-dimensional unitary representation of G and fix an orthonormal basis in H_π. Then there exist $M \in \mathbb{N}$ and a constant C_2 such that each $F \in \mathscr{M}_n(X)$ has a representation of the form*

$$F = A\pi_{\mathscr{M}}^{-1} + \sum_{k=1}^{M} (F_k \pi_{\mathscr{M}}(s_k) - L_{s_k}(F_k)),$$

where $A \in \mathscr{M}_n(\mathbb{C})$, $s_k \in K$, $F_k \in \mathscr{M}_n(X)$, $\|A\| \leqslant C_2 \|F\|$, and $\|F_k\| \leqslant C_2 \|F\|$ for $k = 1, \ldots, M$.

Proof. We begin by observing that each $F \in \mathscr{M}_n(X)$ has a representation of the form

$$F = A + \sum_{k=1}^{M} (H_k - L_{s_k}(H_k)),$$

where $A \in \mathscr{M}_n(\mathbb{C})$, $M = Nn^2$, $s_k \in K$, $H_k \in \mathscr{M}_n(X)$, and $\|H_k\| \leqslant C_1 \|F\|$ for $k = 1, \ldots, M$. Indeed, let $F = (f_{i,j}) \in \mathscr{M}_n(X)$. For any $i,j \in \{1, \ldots, n\}$, $f_{i,j}$ can be written in the form

$$f_{i,j} = a_{i,j} + \sum_{l=1}^{N} (f_{i,j,l} - L_{t_{i,j,l}}(f_{i,j,l})),$$

where $a_{i,j} \in \mathbb{C}$, $f_{i,j,l} \in X$, and $t_{i,j,l} \in K$. We define A to be the matrix $(a_{i,j})$ and $H_{i,j,l}$ to be the matrix with $f_{i,j,l}$ in the entry (i,j) and 0's elsewhere. Of course $\|H_{i,j,l}\| = \|f_{i,j,l}\| \leqslant C_1 \|f_{i,j}\| \leqslant C_1 \|F\|$ and we check at once that

$$F = A + \sum_{l=1}^{N} \sum_{i=1}^{n} \sum_{j=1}^{n} (H_{i,j,l} - L_{t_{i,j,l}}(H_{i,j,l}))$$

as required.

We now proceed to show the lemma. Let $F \in \mathscr{M}_n(X)$. By Lemma 4.1 $F\pi_{\mathscr{M}} \in \mathscr{M}_n(X)$. On account of the preceding observation, we can write $F\pi_{\mathscr{M}} = A + \sum_{k=1}^{M} (H_k - L_{s_k}(H_k))$ with $A \in \mathscr{M}_n(\mathbb{C})$, $s_k \in K$, $H_k \in \mathscr{M}_n(X)$, and $\|H_k\| \leqslant C_1 \|\pi_{\mathscr{M}}\|_\infty \|F\|$ for $k = 1, \ldots, M$. Taking into account Lemma 4.1 we see that

$$\begin{aligned}||A|| &\leqslant || F\pi_{\mathscr{M}}|| + \sum_{k=1}^{M} [||H_k|| + ||L_{s_k}(H_k)||] \\ &\leqslant || F|| \, ||\pi_{\mathscr{M}}||_{\infty} + \sum_{k=1}^{M} 2||H_k|| \leqslant || F|| \, ||\pi_{\mathscr{M}}||_{\infty} + \sum_{k=1}^{M} 2C_1||\pi_{\mathscr{M}}||_{\infty}|| F|| \\ &= (2C_1M||\pi_{\mathscr{M}}||_{\infty} + ||\pi_{\mathscr{M}}||_{\infty})|| F||.\end{aligned}$$

On the other hand, we have

$$F = A\pi_{\mathscr{M}}^{-1} + \sum_{k=1}^{M} (H_k\pi_{\mathscr{M}}^{-1} - L_{s_k}(H_k)\pi_{\mathscr{M}}^{-1}).$$

For every $k = 1, \dots, M$ let $F_k = H_k\pi_{\mathscr{M}}^{-1}\pi_{\mathscr{M}}(s_k)^{-1}$. Lemma 4.1 yields

$$|| F_k|| \leqslant ||H_k|| \, ||\pi_{\mathscr{M}}^{-1}||_{\infty}^{2} \leqslant C_1||\pi_{\mathscr{M}}^{-1}||_{\infty}^{2}||\pi_{\mathscr{M}}||_{\infty}|| F||$$

for each $k = 1, \dots, M$. Let us observe that

$$\begin{aligned}L_{s_k}(F_k)(t) &= F_k(s_k^{-1}t) = H_k(s_k^{-1}t)\pi_{\mathscr{M}}^{-1}(s_k^{-1}t)\pi_{\mathscr{M}}(s_k)^{-1} \\ &= H_k(s_k^{-1}t)\pi_{\mathscr{M}}(t^{-1}s_k)\pi_{\mathscr{M}}(s_k)^{-1} = L_{s_k}(H_k)(t)\pi_{\mathscr{M}}^{-1}(t)\end{aligned}$$

for all $k = 1, \dots, M$ and $t \in G$. We thus get

$$F = A\pi_{\mathscr{M}}^{-1} + \sum_{k=1}^{M} (F_k\pi_{\mathscr{M}}(s_k) - L_{s_k}(F_k)). \qquad \square$$

Lemma 4.3. *Suppose that there exists a constant D_2 such that*

$$|L_t(f)| \leqslant D_2|f|$$

for all $f \in X$ and $t \in K$. If $T \in \mathfrak{A}$ is such that $\dim T(\mathfrak{S}) < \infty$, *then* $T(\mathfrak{S}) = \{0\}$.

Proof. Pick an n-dimensional unitary representation π and fix an orthonormal basis of H_π.

Let $\mathfrak{X}$ denote the matrix space $\mathscr{M}_n(X)$ when equipped with the norm $||\cdot||$ and let $\mathfrak{Y}$ denote the matrix space $\mathscr{M}_n(X)$ when equipped with the norm $|\cdot|$. T is a continuous linear operator from $(X, |\cdot|)$ into itself. We denote its norm by $||T||$. T lifts to a continuous linear operator from $\mathfrak{Y}$ into itself which is easily checked to have the same norm as T. For abbreviation, we continue to write T for this latter operator.

Set

$$\mathfrak{R} = \left\{ T(F) \in \mathfrak{Y} : F \in \mathcal{M}_n(\mathfrak{S}) \text{ and } \int_G T(F)(t)\pi_{\mathcal{M}}(t)\,dt = 0 \right\}.$$

The property that $\dim T(\mathfrak{S}) < \infty$ entails that $\dim \mathfrak{R} < \infty$ and $\mathfrak{R}$ is then closed in $\mathfrak{Y}$ (this is a key fact in the proof of the lemma). Let $\mathfrak{Z}$ be the quotient Banach space $\mathfrak{Y}/\mathfrak{R}$. Let us denote by $|\cdot|_{\mathfrak{Z}}$ the quotient norm on $\mathfrak{Z}$ and let us denote by Q the quotient map from $\mathfrak{Y}$ onto $\mathfrak{Z}$. Let Φ be the identity map from $\mathfrak{X}$ into $\mathfrak{Y}$ and let D_1 be the constant given in Lemma 2.2. On account of Lemma 2.1(ii), it is easily seen that $\mathfrak{S}(\Phi) = \mathcal{M}_n(\mathfrak{S})$.

For every $s \in K$ we define $\Psi_{\pi,s} : \mathfrak{Y} \to \mathfrak{Y}$ by

$$\Psi_{\pi,s}(F) = F\pi_{\mathcal{M}}(s) - L_s(F)$$

for each $F \in \mathfrak{Y}$. The operator $\Psi_{\pi,s}$ satisfies the following properties:

(i) $\Psi_{\pi,s}$ is continuous and $||\Psi_{\pi,s}|| \leqslant D_2 + ||\pi_{\mathcal{M}}||_\infty$. Indeed, for every $F \in \mathfrak{X}$, we have

$$|\Psi_{\pi,s}(F)| \leqslant |F\pi_{\mathcal{M}}(s)| + |L_s(F)| \leqslant |F|\,||\pi_{\mathcal{M}}(s)|| + D_2|F| \leqslant |F|\,||\pi_{\mathcal{M}}||_\infty + D_2|F|.$$

(ii) $\int_G \Psi_{\pi,s}(F)(t)\pi_{\mathcal{M}}(t)\,dt = 0$ for each $F \in \mathfrak{Y}$. This follows from the left-invariance of Haar measure. Indeed,

$$\begin{aligned}
\int_G \Psi_{\pi,s}(F)(t)\pi_{\mathcal{M}}(t)\,dt &= \int_G (F(t)\pi_{\mathcal{M}}(s) - L_s(F)(t))\pi_{\mathcal{M}}(t)\,dt \\
&= \int_G F(t)\pi_{\mathcal{M}}(s)\pi_{\mathcal{M}}(t)\,dt - \int_G L_s(F)(t)\pi_{\mathcal{M}}(t)\,dt \\
&= \int_G F(t)\pi_{\mathcal{M}}(s)\pi_{\mathcal{M}}(t)\,dt - \int_G F(s^{-1}t)\pi_{\mathcal{M}}(t)\,dt \\
&= \int_G F(t)\pi_{\mathcal{M}}(s)\pi_{\mathcal{M}}(t)\,dt - \int_G F(t)\pi_{\mathcal{M}}(st)\,dt = 0.
\end{aligned}$$

(iii) $T\Psi_{\pi,s} = \Psi_{\pi,s}T$ for each $T \in \mathfrak{A}$, because $L_sR_t = R_tL_s$ for each $t \in G$.

(iv) $\Psi_{\pi,s}(\mathfrak{S}(\Phi)) \subset \mathfrak{S}(\Phi)$, which is clear from $\mathfrak{S}(\Phi) = \mathcal{M}_n(\mathfrak{S})$ together with Lemma 2.1(i).

According to (iv), $(T\Psi_{\pi,s})(\mathfrak{S}(\Phi)) \subset T(\mathfrak{S}(\Phi))$. From (ii) and (iii) we now deduce that $(T\Psi_{\pi,s})(\mathfrak{S}(\Phi)) = (\Psi_{\pi,s}T)(\mathfrak{S}(\Phi)) \subset \mathfrak{R}$ and hence that

$$QT\Psi_{\pi,s}\Phi$$

is continuous for each $s \in K$.

On account of Lemma 4.2, for every $F \in \mathfrak{X}$ we have

$$F = A\pi_{\mathscr{M}}^{-1} + \sum_{k=1}^{M} (F_k \pi_{\mathscr{M}}(s_k) - L_{s_k}(F_k)),$$

where $A \in \mathscr{M}_n(\mathbb{C})$, $||A|| \leqslant C_2||F||$, $s_k \in K$, $F_k \in \mathscr{M}_n(X)$, and $||F_k|| \leqslant C_2||F||$ for $k = 1, \ldots, M$. Thus

$$QT\Phi(F) = QT(A\pi_{\mathscr{M}}^{-1}) + \sum_{k=1}^{M} QT\Psi_{\pi,s_k}\Phi(F_k)$$

and so

$$|QT\Phi(F)|_3 \leqslant |QT(A\pi_{\mathscr{M}}^{-1})|_3 + \sum_{k=1}^{M} |QT\Psi_{\pi,s_k}\Phi(F_k)|_3.$$

On the other hand, we have

$$\begin{aligned}|QT(A\pi_{\mathscr{M}}^{-1})|_3 &\leqslant |T(A\pi_{\mathscr{M}}^{-1})| \leqslant ||T|||A\pi_{\mathscr{M}}^{-1}| \\ &\leqslant ||T||\,||A|||\pi_{\mathscr{M}}^{-1}| \leqslant C_2||T||\,||F|||\pi_{\mathscr{M}}^{-1}|.\end{aligned}$$

According to Lemma 2.2 and (i), we have

$$\begin{aligned}|QT\Psi_{\pi,s_k}\Phi(F_k)|_3 &= |(QT\Psi_{\pi,s_k})\Phi(F_k)|_3 \leqslant D_1||QT\Psi_{\pi,s_k}||\,||F_k|| \\ &\leqslant D_1||T||(||\pi_{\mathscr{M}}||_\infty + D_2)||F_k|| \leqslant D_1||T||(||\pi_{\mathscr{M}}||_\infty + D_2)C_2||F||.\end{aligned}$$

Consequently,

$$|QT\Phi(F)|_3 \leqslant (||T||C_2|\pi_{\mathscr{M}}^{-1}| + MD_1||T||(||\pi_{\mathscr{M}}||_\infty + D_2)C_2)||F||$$

for each $F \in \mathfrak{X}$ which implies that $QT\Phi$ is continuous and hence that $T(\mathfrak{S}(\Phi)) \subset \mathfrak{R}$.

We are now in a position to show that $T(\mathfrak{S}) = \{0\}$. Let $f \in \mathfrak{S}$. Since $fI \in \mathfrak{S}(\Phi)$, where I stands for the identity matrix, it follows that $T(fI) \in \mathfrak{R}$. Hence $\int_G T(fI)(t)\pi_{\mathscr{M}}(t)\,dt = 0$. On the other hand, it is clear that the operator $\pi(T(f))$ is represented by the matrix $\int_G T(fI)(t)\pi_{\mathscr{M}}(t)\,dt (= 0)$. Since π is arbitrary, Lemma 2.1(iii) now shows that $T(f) = 0$, as required. □

5. Uniqueness of norm

Denote $L_0^2(G) = \{f \in L^2(G) : \int_G f(t)\,dt = 0\}$. The trivial representation of G is said to be weakly contained in the left regular representation of G on $L_0^2(G)$ if there exists a net (f_α) in $L_0^2(G)$ such that $||f_\alpha||_2 = 1$ and $\lim ||f_\alpha - L_t f_\alpha||_2 = 0$ for each $t \in G$. Such a group is said, in [12], to have the mean-zero weak containment property. Examples of compact groups which do not have the mean-zero weak containment property includes many matrix groups, such as $SO(n)$ for $n \geqslant 3$, and all compact simple Lie groups (see [2,8,12,15]). See also [10,11] for more examples of compact groups which do not have the mean-zero weak containment property.

Lemma 5.1. *If the trivial representation of G is not weakly contained in the left regular representation of G on $L_0^2(G)$ and X is either $L^p(G)$ for $1 < p \leqslant \infty$ or $C(G)$, then there exist $N \in \mathbb{N}$, a constant C, and $t_1, \dots, t_N \in G$ such that each $f \in X$ has a representation of the form*

$$f = a + \sum_{k=1}^{N} (f_k - L_{t_k}(f_k)),$$

where $a \in \mathbb{C}$, $f_k \in X$, and $||f_k|| \leqslant C||f||$ for $k = 1, \dots, N$.

Proof. Throughout the proof, X_0 denotes the closed linear subspace of X defined by $X_0 = \{f \in X : \int_G f(t)\, dt = 0\}$.

We begin by observing that there exists $N \in \mathbb{N}$ and $t_1, \dots, t_N \in G$ such that the map Φ from X_0^N into X_0 defined by

$$\Phi(f_1, \dots, f_N) = \sum_{k=1}^{N} (f_k - L_{t_k}(f_k))$$

for all $f_1, \dots, f_N \in X_0$ is surjective. This fact is shown in [12, Proposition] in the case when $1 < p < \infty$. When X is either $L^\infty(G)$ or $C(G)$ this fact is established in [16].

The open mapping theorem now shows that Φ is open and hence that there is a constant C such that each $f \in X_0$ can be written as $\Phi(f_1, \dots, f_N)$ with $f_k \in X_0$ and $||f_k|| \leqslant C||f||$ for $k = 1, \dots, N$. This clearly gives the required representation of f. Finally, if $f \in X$ let $a = \int_G f(t)\, dt$, so that $f - a \in X_0$, to get the desired representation. $\square$

Theorem 5.1. *Let G be a compact group with the property that the trivial representation of G is not weakly contained in the regular representation of G on $L_0^2(G)$ and let X be either $L^p(G)$ for $1 < p \leqslant \infty$ or $C(G)$. Then every complete norm $|\cdot|$ on X that makes translations from $(X, |\cdot|)$ into itself continuous is equivalent to $||\cdot||_p$ or $||\cdot||_\infty$, respectively.*

Proof. We first observe that Lemma 5.1 shows that G satisfies the requirement in Section 4.

Suppose that the theorem fails to be true. Then Lemma 2.1 shows that $\Delta \neq \emptyset$.

By Lemma 3.5 either Δ is finite or there exists $T \in \mathfrak{A}$ such that the set $\{[\pi] \in \Delta : \pi(T) \neq 0\}$ is nonempty and finite. We shall deal with both cases simultaneously. For the first case we write $T = R_e \in \mathfrak{A}$. Of course, $\pi(T) \neq 0$ for each $[\pi] \in \Delta$. Accordingly, in each of the cases there exists $T \in \mathfrak{A}$ such that $\Sigma = \{[\pi] \in \Delta : \pi(T) \neq 0\}$ is nonempty and finite.

We claim that $\dim T(\mathfrak{S}) < \infty$. Indeed, let us write $\Sigma = \{[\pi_1], \ldots, [\pi_n]\}$ and let $f \in \mathfrak{S}$ be such that $\pi_k(T(f)) = 0$ for $k = 1, \ldots, n$. Note that $\pi(T(f)) = 0$ if $[\pi] \in \Sigma$, that $\pi(f) = 0$ and so $\pi(T(f)) = \pi(f)\pi(T) = 0$ if $[\pi] \in \hat{G} \backslash \Delta$, and finally that $\pi(T) = 0$ and thus $\pi(T(f)) = 0$ if $[\pi] \in \Delta \backslash \Sigma$. We thus obtain that the map $g \mapsto (\pi_1(g), \ldots, \pi_n(g))$ from $T(\mathfrak{S})$ into $\mathscr{L}(H_{\pi_1}) \oplus \cdots \oplus \mathscr{L}(H_{\pi_n})$ is injective. Since $\dim \mathscr{L}(H_{\pi_1}) \oplus \cdots \oplus \mathscr{L}(H_{\pi_n}) < \infty$ we thus get $\dim T(\mathfrak{S}) < \infty$, as claimed.

Let $t_1, \ldots, t_N \in G$ given by the preceding lemma and set $D_2 = \sup\{|L_{t_i}(f)| : f \in X,\ |f| = 1,\ i = 1, \ldots, N\}$. We now apply Lemma 4.3 to obtain that $T(\mathfrak{S}) = 0$.

Finally, take $[\pi] \in \Delta$ with $\pi(T) \neq 0$. We have $\pi(\mathfrak{S})\pi(T) = \pi(T(\mathfrak{S})) = 0$ and Lemma 3.1 gives $\pi(\mathfrak{S}) = 0$, which contradicts the choice of π. $\square$

Remark 5.1. Since the trivial representation of $SO(n)$ is not weakly contained in the left regular representation of $SO(n)$ on $L^2_0(SO(n))$ whenever $n \geqslant 3$, Theorem 5.1 shows that $L^p(SO(n))$ for $1 < p \leqslant \infty$ and $C(SO(n))$ carry a unique (up to equivalence) complete norm that makes translations continuous provided that $n \geqslant 3$. Let us remark that $SO(2) = \mathbb{T}$ and that it is shown in [5] that $L^p(\mathbb{T})$ for $1 < p < \infty$ carries a unique (up to equivalence) complete norm that makes translation continuous, unlike $L^\infty(\mathbb{T})$ and $C(\mathbb{T})$ do not enjoy this property. We finally observe that $SO(1) = \{1\}$ and hence $L^p(SO(1))$ for $1 \leqslant p \leqslant \infty$ and $C(SO(1))$ carry a unique (up to equivalence) norm, because all of them are finite dimensional. Finally, let us point out that we shall prove in Corollary 6.1 that $L^1(SO(n))$ for $n \geqslant 2$ does not have a unique translation invariant norm.

Let us remark that if G and X are as in Theorem 5.1, then every left translation invariant linear functional on X is continuous [12,16]. In fact, it turns out that the uniqueness of norm on $L^p(G)$ is closely related to the classical problem whether or not every translation invariant functional is automatically continuous. As a matter of fact, we obtain the following result.

Theorem 5.2. *Let G be a compact group, $1 < p \leqslant \infty$, and suppose that every left invariant linear functional on $L^p(G)$ is a constant multiple of the Haar integral. Then every complete norm $|\cdot|$ on $L^p(G)$ that makes translations from $(L^p(G), |\cdot|)$ into itself continuous and that makes the map $t \mapsto L_t$ from G into $\mathscr{L}(L^p(G), |\cdot|)$ bounded is equivalent to $\|\cdot\|_p$.*

Proof. Let us first note that [13, Theorem 2] shows that G satisfies the requirement in Section 4. By hypothesis there exists a constant D_2 such that $|L_t(f)| \leqslant D_2 |f|$ for all $f \in L^p(G)$ and $t \in G$.

Now the proof runs as before. $\square$

Remark 5.2. Let us remark that if $|\cdot|$ is a complete norm on $L^p(G)$ with the property that left translations from $(L^p(G), |\cdot|)$ into itself are continuous and that the map $t \mapsto L_t f$ from G into $(L^p(G), |\cdot|)$ is continuous for each $f \in L^p(G)$, then the map $t \mapsto L_t$ from G into $\mathscr{L}(L^p(G), |\cdot|)$ is bounded. Indeed, the continuity of the map $t \mapsto L_t f$ from G into $(L^p(G), |\cdot|)$ and the compactness of G imply that the map is bounded for each $f \in L^p(G)$. The uniform boundedness theorem now shows that the subset $\{L_t : t \in G\}$ of $\mathscr{L}(L^p(G), |\cdot|)$ is bounded.

Remark 5.3. On account of [13, Theorem 2], if $1 < p \leqslant \infty$ and the group G is such that there is a left invariant linear functional on $L^p(G)$ which is not a constant multiple of the Haar integral, then the codimension of the linear subspace E of $L^p(G)$ generated by $\{f - L_t(f) : f \in L^p(G),\ t \in G\}$ is uncountable. Hence there exists a discontinuous linear functional ϕ on $L^p(G)$ that vanishes on E. We thus get a discontinuous left invariant linear functional on $L^p(G)$.

6. Nonuniqueness of norm

In this section we prove that translations do not determine the norm neither of $L^p(G)$ for $1 \leqslant p \leqslant \infty$ nor of $C(G)$ provided that there is a discontinuous two-sided invariant linear functional. Discontinuous translation invariant linear functionals have been shown to exist on a wide variety of translation invariant spaces of functions on locally compact groups.

Theorem 6.1. *Let G be a compact group and let X be either $L^p(G)$ for $1 \leqslant p \leqslant \infty$ or $C(G)$. If there exists a discontinuous two-sided invariant linear functional on X, then there exists a complete norm $|\cdot|$ on X that makes translations from $(X, |\cdot|)$ into itself continuous, that makes the maps $t \mapsto L_t$ and $t \mapsto R_t$ from G into $\mathscr{L}(X, |\cdot|)$ bounded, and that is not equivalent to $\|\cdot\|$.*

Proof. Let ϕ be a discontinuous two-sided invariant linear functional on X. We claim that there is a discontinuous two-sided invariant linear functional ψ on X such that $\psi(1) = 1$. Indeed, if $\phi(1) = \alpha \neq 0$ then we take $\psi = \alpha^{-1}\phi$. If $\phi(1) = 0$, then we define $\psi(f) = \phi(f) + \int_G f(t)\,dt$ for each $f \in X$.

Since the map $f \mapsto 2f - \psi(f)1$ is a linear bijection from X onto itself, it may be concluded that $|f| = \|2f - \psi(f)1\|$ is a complete norm on X that is not equivalent to $\|\cdot\|$. Let $t, s \in G$. For every $f \in X$ we have

$$\begin{aligned} |L_t f| &= \|2L_t(f) - \psi(L_t(f))1\| = \|L_t(2f) - \psi(f)1\| = \|L_t(2f - \psi(f)1)\| \\ &\leqslant \|L_t\|\,\|2f - \psi(f)1\| = \|L_t\||f|, \end{aligned}$$

which shows that $L_t \in \mathscr{L}(X, |\cdot|)$ and that the map $t \mapsto L_t$ from G into $\mathscr{L}(X, |\cdot|)$ is bounded. In the same manner we can see that $R_t \in \mathscr{L}(X, |\cdot|)$ for each $t \in G$ and that the map $t \mapsto R_t$ from G into $\mathscr{L}(X, |\cdot|)$ is bounded. $\square$

By Saeki [13, Theorem 1], for every infinite compact group G, there exists a discontinuous two-sided invariant linear functional on $L^1(G)$. On account of the above theorem, we have the following result about the nonuniqueness of norm on $L^1(G)$.

Corollary 6.1. *Let G be an infinite compact group. Then there exists a complete norm* $|\cdot|$ *on* $L^1(G)$ *that makes translations from* $(L^1(G), |\cdot|)$ *into itself continuous, makes the maps* $t \mapsto L_t$ *and* $t \mapsto R_t$ *from G into* $\mathscr{L}(L^1(G), |\cdot|)$ *bounded, and it is not equivalent to* $\|\cdot\|_1$.

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